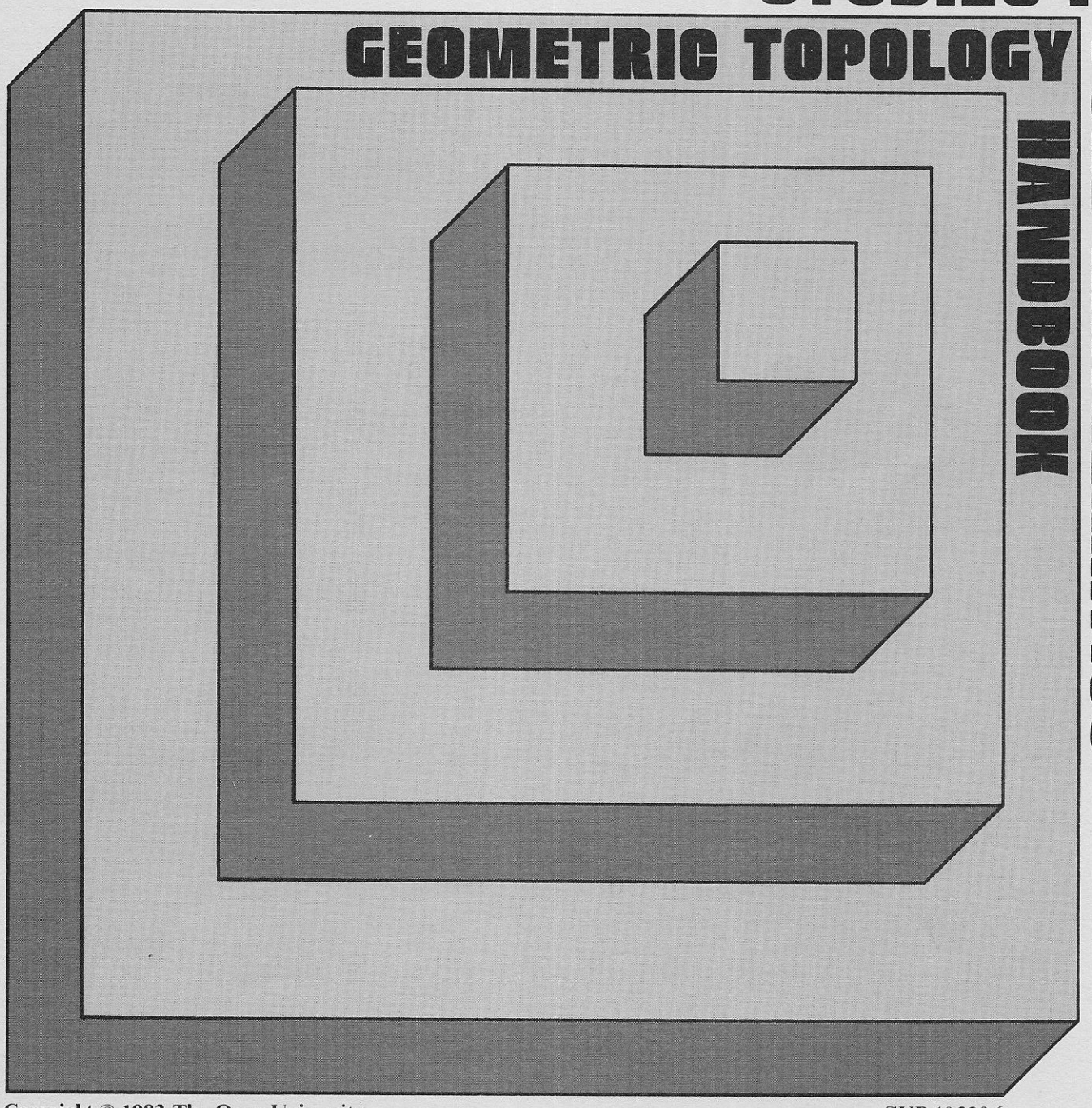
THE OPEN UNIVERSITY
Mathematics: A Third Level Course
M335 Studies in Pure Mathematics

Note that you may take this handbook into the examination at the end of the course, provided you have not made any written additions to it other than corrections.

STUDIES IN PURE MATHEMATICS

GEOMETRIC TOPOLOGY

HANDBOOK



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Unit 1 Surfaces

Polar coordinates

Polar coordinates (r, θ) in the plane are related to rectangular Cartesian coordinates (x, y) by the formulae

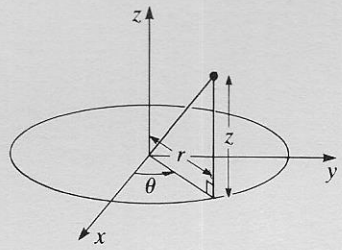
$$x = r\cos\theta,$$
$$y = r\sin\theta.$$

B p. 3

Cylindrical coordinates

Replacing x and y in the Cartesian coordinates (x, y, z) by polar coordinates gives cylindrical coordinates (r, θ, z) (also known as cylindrical polar coordinates).

B pp. 3, 4



Stereographic projection

The one-to-one correspondence of the unit sphere $x^2 + y^2 + z^2 = 1$ minus the "north pole" $N = (0, 0, 1)$, with the equatorial plane $z = 0$, which maps the point P of the sphere to the point P' where the line PN meets the plane $z = 0$.

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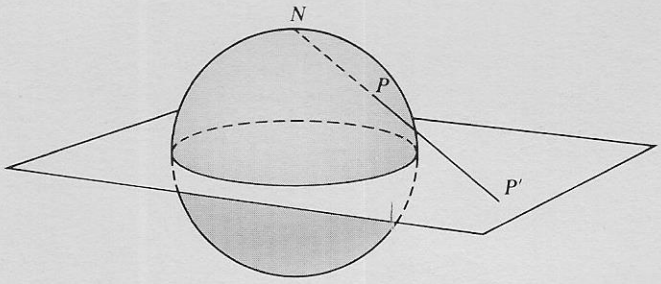


Homeomorphism

A continuous function $f: X \longrightarrow Y$ with an inverse function $f^{-1}: Y \longrightarrow X$ which is also continuous.

Unit 1, Section 1.3
B pp. 12, 13

Topological sphere

Any set homeomorphic to the standard sphere in $\mathbb{R}^3$, $\{(x, y, z): x^2 + y^2 + z^2 = 1\}$.

B p. 15

Topological torus

Any set homeomorphic to a standard torus (e.g. the one obtained by rotating a circle radius 1 and centre (0, 2, 0) lying in the yz-plane about the z-axis).

B p. 14

Path-connected surface (or set)

A surface S (or, in general, any set) is path-connected if, given any two points A, B of S, there is a path in S from A to B; that is, there is a continuous function, or path,

$$f: [0, 1] \longrightarrow S \quad \text{with } f(0) = A \text{ and } f(1) = B.$$

Unit 1, Section 1.3

Closed rectangle

A closed rectangle in $\mathbb{R}^2$ is a set

$$\{(x, y): a \leq x \leq b,\ c \leq y \leq d\}.$$

Unit 1, Section 1.3

Closed set

A set $X \subseteq \mathbb{R}^2$ is closed if every point c *not* in X can be "closed off" by a closed rectangle containing c in its inside (not on the boundary) which does not meet X.

Unit 1, Section 1.3

Compact set

A set which is both closed and bounded.

Unit 1, Section 1.3

Neighbourhood

A subset of a surface S is a neighbourhood of $P \in S$ if it contains all the points of S within a small distance of P. (The neighbourhood is allowed to contain other points as well.)

Locally disc-like A set S is "locally disc-like" if, for every point P of S, we can find a homeomorphism, f_P, from a neighbourhood of P to the standard open disc $D = \{(x, y): x^2 + y^2 < 1\}$ carrying P to $(0, 0)$. *Unit 1*, Section 1.4

Surface without boundary A set S is a surface without boundary if: *Unit 1*, Section 1.4

(i) S is locally disc-like;

(ii) S is path-connected;

(iii) S is homeomorphic to a compact subset of $\mathbb{R}^n$.

Surface with boundary As above, replacing (i) by

(i)′ for each point P we can find

either (a) a homeomorphism f_P from a neighbourhood of P to the standard open disc $D = \{(x, y): x^2 + y^2 < 1\}$, carrying P to $(0, 0)$,

or (b) a homeomorphism g_P from a neighbourhood of P to the halfdisc $\{(x, y): x^2 + y^2 < 1, y \geq 0\}$, carrying P to $(0, 0)$.

Subdivision of a surface A subdivision of a surface is a dissection of the surface into vertices, edges and faces satisfying the following. *Unit 1*, Section 1.4

1. Each edge starts or ends at a vertex.
2. Edges cross or meet only at vertices.
3. The edges form a path-connected network.
4. The faces are topological discs (homeomorphic to the standard disc).

Boundary of a surface The subset of points P of the surface where *no* homeomorphism exists of the type f_P given in the alternative condition (i)′ (a) above, so that only a homeomorphism g_P of a neighbourhood with a halfdisc exists.

Euler characteristic The Euler characteristic of a (subdivided) surface S with F faces, E edges and V vertices is $\chi(S) = F - E + V$.

Theorem (Not proved in the course) *Unit 2*, Section 2.2
The boundary of a surface consists of a finite number of disjoint curves, each homeomorphic to a circle. Each curve is known as a *boundary curve* of the surface, and the number of boundary curves is a topological invariant.

Classification Theorem A surface is determined up to homeomorphism once we know:

its Euler chacteristic;

how many boundary curves it has;

whether or not it is orientable ("orientable" is defined in *Unit 2*).

Unit 2 Subdivisions and Orientability

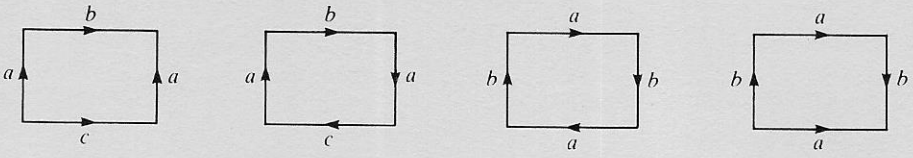
Triangulation A triangulation of a surface is a subdivision (see above) also satisfying the following. *Unit 2*, Section 2.1

1. Every face has exactly 3 edges and 3 vertices.
2. Every edge has exactly 2 vertices.
3. There is at most one face with 3 given vertices.
4. There is at most one edge with 2 given vertices.

Regular subdivision A subdivision in which each face has the same number of edges, and each vertex is at the end of the same number of edges. *Unit 2*, Section 2.1

Cylinder (or annulus)
Möbius band
Projective plane
Klein bottle

Surfaces which can be made by identifying edges of a rectangle, as follows. *Unit 2*, Section 2.3



Cylinder Möbius band Projective plane Klein bottle

Crosscap A name for a particular representation of the Möbius band in $\mathbb{R}^2$ which has self-intersections. *Unit 2*, Section 2.3

Orientation An *orientation* of a surface is any process or rule which defines a consistent sense of rotation over the whole surface. *Unit 2*, Section 2.4

Orientable and non-orientable A surface is called *orientable* if it can be given an orientation, and *non-orientable* otherwise. *Unit 2*, Section 2.4

enting a face of a subdivision To specify an orientation of a face of a subdivision, it is necessary only to specify a direction of travel around the boundary. *Unit 2*, Section 2.4

Compatibility of ntation of faces In a subdivided surface, we say that the orientations of two faces having a common edge b are compatible across b if the "arrows" showing the direction of travel round the boundaries of the two faces are in opposite directions on the two sides of b. *Unit 2*, Section 2.4

Orientation of a bdivided surface An orientation of a subdivided surface is an orientation of each face which satisfies the compatibility condition above. *Unit 2*, Section 2.4

Theorem A surface is non-orientable if and only if it contains a Möbius band. *Unit 2*, Section 2.4

Theorem A non-orientable surface without boundary cannot be embedded in $\mathbb{R}^3$. *Unit 2*, Section 2.4

Unit 3 Combinatorial Surfaces

Let D be a disc with n evenly spaced vertices $p_1, \ldots, p_n$ which divide the boundary circle of D into n arcs $a_1, \ldots, a_n$.

(n-sided) logical polygon An (n-sided) topological polygon π is the image of D under a one-to-one continuous function f. The points $f(P_i)$ are called the *vertices* of π, and the (curved) lines $f(a_i)$ are called the *edges* of π.

Singular logical polygon A singular topological polygon is the image of D under a continuous function f which is one-to-one except possibly on the boundary of D. For each edge a_i of D, there are two possibilities. For each point P of a_i other than the two vertices P_i, P_{i+1} (interpret P_{n+1} as P_1):

(i) there is no other point Q of D with $f(P) = f(Q)$;

or

(ii) there is exactly one point P' with $f(P') = f(P)$, and in this case as P moves from P_i to P_{i+1}, P' also traces out another edge a_j, either from P_j to P_{j+1} or from P_{j+1} to P_j.

We say the edge $b_i = f(a_i)$ is identified with the edge b_j (if P' traces out a_j from P_j to P_{j+1}) or with b_j^{-1} (if P' traces out a_j from P_{j+1} to P_j).

Edge equation

The edge equation of a (singular) topological polygon is obtained from $b_1, \ldots, b_n = 1$ as follows.

Replace b_j with b_i if b_j is identified with b_i for $i < j$, and replace b_j with b_i^{-1} if b_j^{-1} is identified with b_i for $i < j$.

Combinatorial representation of a surface

A system of edge equations is a combinatorial representation of a surface such that:

B p. 61

(i) no letter appears more than twice;

(ii) no matter how the system is divided into two sets of equations, at least one letter appears in an equation of each set.

Trivial equivalence of edge equations

An equivalence achieved by a sequence of operations of types 1 and 2 below.

1. An initial (terminal) letter transposed to become the initial (terminal) letter on the opposite side of an equation, for example
$$aB = C \longrightarrow B = a^{-1}C \quad \text{or} \quad Ba = C \longrightarrow B = Ca^{-1},$$
where B, C stand for any blocks of letters.
2. A new letter, not appearing in the equations, is substituted for an old letter wherever the old letter appears.

"Cut and paste" operations or equivalences

1. Select a new letter, x, and replace an equation $ABC = 1$ by the two equations $AxC = 1$ and $x = B$ (where A, B, C stand for any blocks of letters).
2. Select a new letter, x, and insert the combination xx^{-1} (or $x^{-1}x$) in any equation in any position.
3. Select new letters, u, v, and replace an existing letter x with uv (and x^{-1} by $v^{-1}u^{-1}$, if x^{-1} occurs).
4. The inverses of 1, 2, 3.

Unit 3, Section 3.

Combinatorial equivalence

Two systems of edge equations are combinatorially equivalent if one system can be transformed into the other by a sequence of trivial equivalences and "cut and paste" equivalences, in any order.

Unit 3, Section 3.

Theorem

(Not proved in the course)
Combinatorially equivalent surfaces are homeomorphic.

Models of $\mathbb{RP}^2$

There are various ways of constructing the projective plane. All the following constructions, or "models", are homeomorphic.

Geometer's "plane" model

The "points" are

(i) ordinary points of $\mathbb{R}^2$,

or

(ii) "ideal" points—one ideal point for every direction in $\mathbb{R}^2$.

Geometer's "ray" model

The "points" are lines through the origin in $\mathbb{R}^3$.

"Sphere" model

The "points" are *pairs* of antipodally opposite points on S^2, the standard unit sphere.

"Hemisphere" model

The southern hemisphere of S^2, with the antipodally opposite points on the equator identified.

"Topologist's" model

A square with the edges identified according to the edge equation $abab = 1$.

Theorems

1. The projective plane is non-orientable and consists of a Möbius band glued to a disc along their boundary curves ∂:
$$\mathbb{RP}^2 = M^2 \cup_\partial D^2.$$
2. A projective line is a topological circle, and since it is the central circle of a Möbius band, it is orientation-reversing. A conic is also a topological circle, but since it is the central circle of an annulus, it is not orientation-reversing.

Unit 4 Canonical Equations and Connected Sums of Surfaces

Handle When an edge equation has a block of symbols of the form $aba^{-1}b^{-1}$, we refer to this as a *handle*. It corresponds geometrically to a torus with a hole or "handle". *Unit 4*, Section 4.1 See also Section 4.3

Crosscap A block of symbols aa in an edge equation *or* a particular representation of the Möbius band which corresponds geometrically to such a block.

Cuff A block of symbols ded^{-1} in an edge equation.

Canonical equation An edge equation of the form *Unit 4*, Section 4.2

$$\underbrace{(a_1b_1a_1^{-1}b_1^{-1}) \ldots (a_pb_pa_p^{-1}b_p^{-1})}_{p \text{ blocks}} \underbrace{d_1e_1d_1^{-1} \ldots d_re_rd_r^{-1}}_{r \text{ blocks}} = 1$$

(orientable canonical form),

or

$$\underbrace{a_1a_1 \ldots a_qa_q}_{q \text{ blocks}} \underbrace{d_1e_1d_1^{-1} \ldots d_re_rd_r^{-1}}_{r \text{ blocks}} = 1$$

(non-orientable canonical form).

Genus

(a) *Orientable surface*: the number of handles in the canonical equation.

(b) *Non-orientable surface*: the number of crosscaps in the canonical equation.

Vertex symbol (xy) Given an edge equation $AxyB = 1$, the vertex symbol (xy) is used for the common vertex of the two edges x and y, i.e. the terminal vertex of x and the initial vertex of y. Similarly, $(v^{-1}w)$ is the common vertex of v^{-1} and w, i.e. the terminal vertex of v^{-1} (=initial vertex of v) and the initial vertex of w.

Connected sum of two surfaces To form the connected sum $L \# M$ of two surfaces L and M, remove an (open) disc from each and glue L to M along the circular boundaries of the holes thus created. *Unit 4*, Section 4.3

Lemmas on symbol manipulation

1. If a polygon of a surface has an equation *Unit 4*, Section 4.1
$$AxBCxD = 1,$$
it is combinatorially equivalent to the surface with this equation replaced by
$$AxCxB^{-1}D = 1,$$
and vice versa.
2. Similarly, combinatorial equivalence permits an equation of the form $AxBCx^{-1}D = 1$ to be replaced by $AxCBx^{-1}D = 1$.
3. Any system of edge equations is combinatorially equivalent to a system with a single equation. Hence, any surface may be represented by a single equation.

Theorem S is non orientable in the *Unit 2* sense $\iff$ S is represented by a single edge equation of the form $\ldots a \ldots a \ldots = 1$. *Unit 4*, Section 4.2

Theorem Orientability, Number of boundary curves, Euler characteristic — are all preserved by combinatorial equivalence. *Unit 4*, Section 4.2

Theorem Every system of edge equations is combinatorially equivalent to one of the canonical quations. Every surface is combinatorially equivalent to, and hence homeomorphic to, one of the surfaces given by a canonical equation. More explicitly, an orientable surface must be a sphere with p handles and r cuffs, and a non-orientable surface must be a sphere with q crosscaps and r cuffs. *Unit 4*, Section 4.2

Theorem $\mathbb{RP}^2$ minus an open disc is homeomorphic to a Möbius band, M^2: *Unit 4*, Section 4.3

$$\mathbb{RP}^2 \backslash \mathring{D}^2 \cong M^2.$$

Theorem The Klein bottle, K^2, is homeomorphic to two Möbius bands glued along their boundaries:

$$K^2 \cong M^2 \cup_\partial M^2.$$

Theorem

$$3\,\mathbb{RP}^2 \cong T^2 \# \mathbb{RP}^2$$

Theorem For any surface S,

$$S \# S^2 \cong S, \qquad S \# D^2 \cong S \backslash \mathring{D}^2.$$

Theorem An orientable surface is either the sphere S^2 or else is homeomorphic to the connected sum of p tori and r discs; in symbols, $S = S^2$ or $S \cong pT^2 \# rD^2$. *Unit 4*, Section 4.3

A non-orientable surface S is homeomorphic to the connected sum of q projective planes and r discs; in symbols, $S \cong q\mathbb{RP}^2 \# rD^2$.

Theorem If two surfaces, L, M, have canonical equations $A = 1$, $B = 1$, respectively, then $AB = 1$ is an edge equation for $L \# M$. *Unit 4*, Section 4.3

Theorem (Not proved in the course)
Every surface can be triangulated.

Theorem (Not proved in the course)
The Euler characteristic is a topological invariant of a surface; that is, if L, M are homeomorphic surfaces, then $\chi(L) = \chi(M)$.

Unit 5 Complex Loci and the Riemann Sphere

Complex locus The complex locus $f(z, w) = 0$ is the set of pairs of complex numbers (z, w) which satisfy the equation. *Unit 5*, Section 5.1

Riemann sphere The plane of complex numbers plus a "point at infinity", usually denoted by ∞. The Riemann surface itself is denoted by $\mathbb{C} \cup \{\infty\}$. Stereographic projection gives a one-to-one correspondence from the sphere S^2 to $\mathbb{C} \cup \{\infty\}$, with the north pole N corresponding to ∞. *Unit 5*, Section 5.1

"Infinite points" of a complex locus Given a complex locus $f(z, w) = 0$, where f is a polynomial in z and w, we say: *Unit 5*, Section 5.1

(i) a point (∞, w), $w \neq \infty$, satisfies $f(z, w) = 0$ if and only if $(\tilde{z}, w) = (0, w)$ satisfies the transformed equation $f\left(\frac{1}{\tilde{z}}, w\right) = 0$;

(ii) a point (z, ∞), $z \neq \infty$, satisfies $f(z, w) = 0$ if and only if $(z, \tilde{w}) = (z, 0)$ satisfies the transformed equation $f\left(z, \frac{1}{\tilde{w}}\right) = 0$;

(iii) the point (∞, ∞) satisfies $f(z, w) = 0$ if and only if $(\tilde{z}, \tilde{w}) = (0, 0)$ satisfies the transformed equation $f\left(\frac{1}{\tilde{z}}, \frac{1}{\tilde{w}}\right) = 0$.

In all these cases, the transformed equation is obtained by substituting $\frac{1}{\tilde{z}}$ or $\frac{1}{\tilde{w}}$ for z or w (or both), and, if necessary, multiplying the result by a power of $\tilde{z}$ or $\tilde{w}$ to get a polynomial equation. Points of the form (z, ∞), (∞, w), (∞, ∞), are called infinite points of the locus.

Complex conic A complex conic is the complex locus $f(z, w) = 0$, where the polynomial f has degree 2 in z and w, i.e.

Unit 5, Sections 5.2, 5.3

$$f(z, w) = Az^2 + Bzw + Cw^2 + Dz + Ew + F,$$

where A, B, C, D, E, F are complex numbers, not all of A, B and C are zero, and z, w are allowed to take complex values.

Non-degenerate complex conic $f(z, w) = 0$ is *degenerate* if f factors into two linear polynomials, and is non-degenerate otherwise.

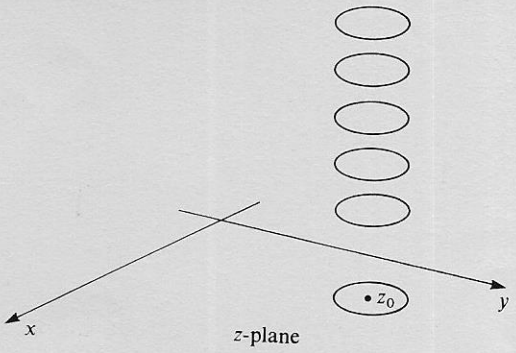
Unit 5, Section 5.3

***n*-fold cover** Given a complex locus

Unit 5, Section 5.1

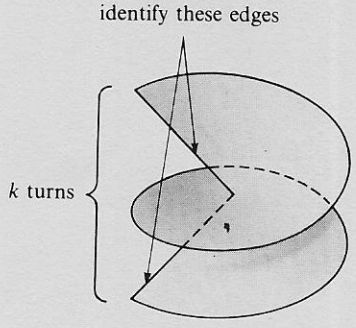
$$f(z, w) = a_0(z)w^n + a_1(z)w^{n-1} + \cdots + a_n(z) = 0$$

(where each $a_i(z)$ is a polynomial in z), then, for each fixed z_0, $f(z_0, w)$ is a polynomial in w of degree at most n. Suppose, for a particular value of z_0, $f(z_0, w) = 0$ has n distinct roots $w(z_0)$.

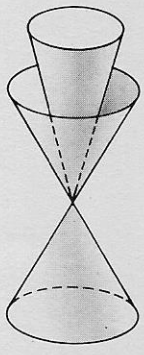


If z_0 varies over a small disc in the z-plane, each of the values of $w(z_0)$ varies over a small disc, which we draw in $\mathbb{R}^3$ as if they are sitting over the patch containing z_0. We say the locus is an *n-fold cover* if $f(z_0, w) = 0$ has exactly n roots, except possibly at a finite number of points z_0.

Pinch points, Branch points As z varies over the whole of $\mathbb{C} \cup \{\infty\}$, the discs referred to in the definition of an n-fold cover (given above) join up at the exceptional points z_0 where $w(z_0)$ has *fewer* than n distinct values. Two ways in which they join up for the special values of z are *pinch points* and *branch points*.



Branch points



Pinch points

Note that, strictly speaking, branch points and pinch points are points on the *locus*, not the values of z where they occur.

Riemann surface Suppose $f(z, w) = Az^2 + Bzw + Cw^2 + Dz + Ew + F = 0$ is the equation of a non-degenerate complex conic. For a fixed value of z, the equation has two distinct roots, except where

$$(Bz + E)^2 - 4C(Az^2 + Dz + F) = 0. \qquad (*)$$

Unit 5, Section 5.3

B pp. 90–3

Three cases occur.

1. $C = 0$. There is only one value of w for each value of z, and the Riemann surface is the locus itself (plus the infinite point).
2. $B^2 - 4AC \neq 0$. The equation $(*)$ has two distinct roots, t_1, t_2. If we cut $\mathbb{C} \cup \{\infty\}$ along an edge a from ∞ to t_1, and along an edge b from ∞ to t_2, then the Riemann surface consists of two sheets, each equal to the cut sphere, joined according to the edge equations
$$a_1 a_2^{-1} b_1 b_2^{-1} = 1, \quad a_2 a_1^{-1} b_2 b_1^{-1} = 1,$$
where the suffix 1 refers to the first sheet and the suffix 2 refers to the second sheet.
3. $B^2 - 4AC = 0$. The equation $(*)$ has one root, t_1, and the Riemann surface consists of two sheets, each obtained by cutting the Riemann sphere from t_1 to ∞ along a and joining the sheets according to the equations $a_1 a_2^{-1} = 1$, $a_2 a_1^{-1} = 1$.

Theorem A non-degenerate complex conic

$$Az^2 + Bzw + Cw^2 + Dz + Ew + F = 0$$

(plus its infinite points) is a sphere if $A = 0$ or $C = 0$ or $B^2 - 4AC = 0$, and is a pinched sphere otherwise. The branch points and pinch points occur as follows.

Unit 5, Section 5.3

1. $A = 0$, $C \neq 0$. There are two branch points, one of which is (∞, ∞) if $B = 0$, and no pinch points.
2. $C = 0$. No branch points, no pinch points.
3. $A \neq 0$, $C \neq 0$, $B^2 - 4AC = 0$. Two branch points, one of which is (∞, ∞).
4. Two branch points (not at infinity), and a pinch point at (∞, ∞).

Unit 6 Covering Spaces

Local homeomorphism A continuous function $p: \tilde{S} \longrightarrow S$ between two surfaces is a local homeomorphism if every $\tilde{P} \in \tilde{S}$ has a (small) disc $D_{\tilde{P}}$ round $\tilde{P}$ with the property that p restricted to $D_{\tilde{P}}$ is a homeomorphism.

Unit 6, Section 6.1

Regular n-fold covering surface (also called an unbranched n-fold covering surface) If we have a local homeomorphism $p: \tilde{S} \longrightarrow S$ between two surfaces, and the inverse image of every $P \in S$ consists of exactly n points of $\tilde{S}$, we say that $p: \tilde{S} \longrightarrow S$ is a regular (or unbranched) n-fold covering surface. S is called the base surface and $\tilde{S}$ is often (loosely) referred to as the covering surface, or sometimes the "total space".

n-sheeted covering surface Starting with any set of edge equations representing a surface S, make n copies of S and its equations, called the *sheets* of the covering, by writing down the original equations n times, with subscripts 1 to n added to the edge symbols to denote the copy of S to which it belongs. If an edge symbol occurs twice in the original set of equations, permute the subscripts on the second occurrences of the edge symbol in the n copies, in any manner, provided that the combined sets of equations result in a surface $\tilde{S}$. This joins up the n sheets of the covering to make the surface $\tilde{S}$. $\tilde{S}$ is called an *n-sheeted covering surface of S.*

Unit 6, Section 6.2

Riemann surface If S is the Riemann sphere, $\tilde{S}$ is called a *Riemann surface.*

Order of a branch point = branch point deficit

Suppose $p:\tilde{S} \longrightarrow S$ is an n-sheeted covering surface and $\tilde{P}$ is a branch point of $\tilde{S}$. Then at $\tilde{P}$, $k > 1$ sheets of the covering surface are joined together so that instead of one point over $P = p(\tilde{P})$ for each of the k sheets, there is only the point $\tilde{P}$. Then the deficiency in the number of covering points $k - 1$ is called the *order* of the branch point $\tilde{P}$.

Unit 6, Sections 6.2, 6.3

Total branch deficit

The total of the orders (or deficits) of each branch point.

Unit 6, Section 6.2

Theorem

Suppose $p:\tilde{S} \xrightarrow{n} S$ is an n-sheeted (n-fold) covering surface; then

Unit 6, Section 6.2

(i) if S has β boundary curves, $\tilde{S}$ has at least β and at most $n\beta$ boundary curves—in particular, if $\tilde{S}$ is closed (no boundary curves) then so is S, and vice versa; Section 6.3

(ii) if S is orientable, so is $\tilde{S}$; Section 6.2

(iii) $\chi(\tilde{S}) = n\chi(S) - \delta$, where δ is the total branch deficit—in particular, $\chi(\tilde{S}) = n\chi(S)$ for an *unbranched* cover. Section 6.2

Unit 7 Winding Numbers and Brouwer Degree

Loop

A loop in the plane (in a surface S) is a continuous function $f\colon [a, b] \longrightarrow \mathbb{R}^2(S)$.

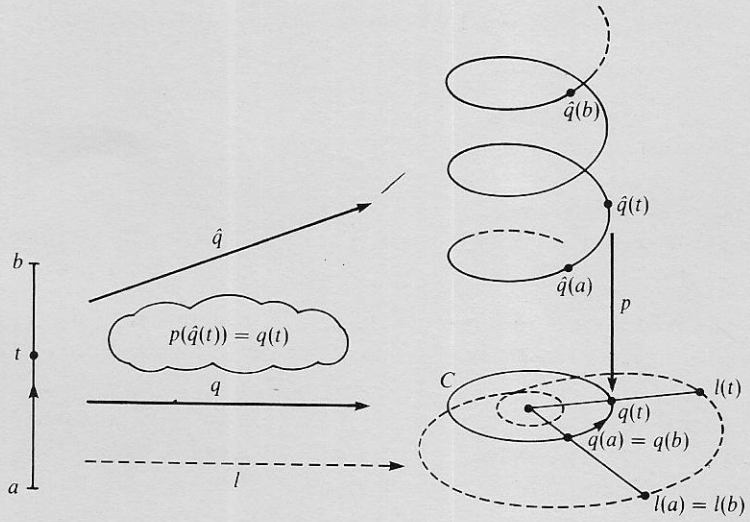
Unit 7, Section 7.1

Covering space of a circle

The continuous function $p\colon \mathbb{R} \longrightarrow C$ exhibits $\mathbb{R}$ as a covering space of the circle C if p is a local homeomorphism; that is, if every $x \in \mathbb{R}$ has a small interval about x on which p is a homeomorphism.

Winding number

In the diagram, l is a loop, $\mathbb{R}$ is a spiral with unit distance between successive turns, and $p\colon \mathbb{R} \longrightarrow C$ is a covering space of the circle C.



Define a function $q\,[a, b] \longrightarrow C$, the "circle of directions" centre P, by letting $q(t)$ be the point where the ray (halfline) from P to $l(t)$ cuts C. Choose a point $\hat{q}(a)$ in the spiral $\mathbb{R}$ which projects to $q(a)$, and lift q to a continuous map $\hat{q}\colon [a, b] \longrightarrow \mathbb{R}$ so that $p(\hat{q}(t)) = q(t)$ for all t in $[a, b]$. Then the winding number of l about P is the vertical distance between $\hat{q}(a)$ and $\hat{q}(b)$.

Homotopy of loops

The loop l_1 is homotopic to the loop l_2 if we can find a family of loops l_v, one loop l_v for each v between 1 and 2, such that the family gives us a continuous variation of l_1 into l_2. That is, if v and w are close, the image points of l_w and l_v are close; in other words, for each point t in $[a, b]$, $l_v(t)$ is close to $l_w(t)$.

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Multiplicity Suppose S, T are orientable surfaces, and $f: S \longrightarrow T$ is a continuous function with the additional property that only a finite number of points of S are mapped to any one point of T (that is, $f^{-1}(Q)$ is a finite set of points of S for every $Q \in T$). The multiplicity of a point $P \in S$ is the winding number of $f(l)$ about $f(P)$, where l is a small loop winding once positively round P. ("Small" means l contains no points of S mapping to $f(P)$ other than P itself.)

(This is also known as the *geometric multiplicity* of P, to distinguish it from the algebraic multiplicity defined later.)

Brouwer degree See Brouwer Degree Theorem.

Argument of a complex number z The complex number $z = x + iy$ (x, y real numbers) can be written in the form $z = r(\cos\theta + i\sin\theta)$. θ is called the angular coordinate or argument of z. It is written $\arg z$ for short.

Variation in $\arg z$ on a loop l If $l: [a, b] \longrightarrow \mathbb{C}$ is a curve (not necessarily a loop) in the plane of complex numbers $\mathbb{C}$, and can be written in the form

$$l(t) = r(t)[\cos\theta(t) + i\sin\theta(t)],$$

where r, θ are *continuous* functions of t, then the variation in $\arg z$ on l is given by $\text{var}(\arg l(t)) = \theta(b) - \theta(a)$.

Zero and pole of a rational function Suppose $f: \mathbb{C} \longrightarrow \mathbb{C}$ is a rational function, that is, $f(z) = p(z)/q(z)$ where p, q are polynomials. Assume p, q have no common factor. Then a *zero* of f is a point where $p(z) = 0$, and a *pole* of f is a point where $q(z) = 0$. These terms can be extended to functions $f: \mathbb{C} \cup \{\infty\} \longrightarrow \mathbb{C} \cup \{\infty\}$; a zero of f is then a point which maps to 0 and a pole is a point which maps to ∞. *Unit 7*, Section 7.4

Algebraic multiplicity of a zero (pole) If $f(z) = p(z)/q(z)$, where p, q are polynomials with no common factors, then the *algebraic multiplicity* of a zero a of f is the integer r where $p(z) = (z - a)^r \bar{p}(z)$ and $\bar{p}(a) \neq 0$. A similar definition applies for a pole, with q replacing p.

Theorem If l_1 and l_2 are homotopic loops, and P does not lie on any of the intermediate loops l_n of the homotopy, then l_1 and l_2 have the same winding number about P. *Unit 7*, Section 7.1

Theorem Given a loop in the plane, the loop divides the plane into regions such that any point in a region can be reached from any other point in the same region without crossing the loop. Then *Unit 7*, Section 7.1

1. the winding numbers of the loop about two points in the same region are the same;
2. the winding number increases by one when crossing a loop segment traced out from left to right.

Theorem (Brouwer Degree Theorem) Given a continuous function $f: S \longrightarrow T$ between closed orientable surfaces such that every point Q in T has a finite number of pre-images (points P mapping to Q), define *Unit 7*, Section 7.1

$$N_Q = \text{sum of multiplicities of all pre-images of } Q.$$

Then N_Q is the same for every point Q of T. This common value is called the Brouwer degree of f.

Theorem If $f: [a, b] \longrightarrow \mathbb{C}$ is a loop, and continuous functions r, θ are given so that *Unit 7*, Section 7.3

$$l(t) = r(t)[\cos\theta(t) + i\sin\theta(t)],$$

then the winding number of l about 0 is $\dfrac{1}{2\pi}(\theta(b) - \theta(a))$.

Theorem Suppose $f, g: [a, b] \longrightarrow \mathbb{C}$ are loops, and the loop $l: [a, b] \longrightarrow \mathbb{C}$ is defined by the formula $h(t) = f(t)g(t)$. Then, about any point not on the loops, *Unit 7*, Section 7.3

$$\omega(h) = \omega(f) + \omega(g).$$

Theorem (Rouché) Suppose l is a loop in $\mathbb{C}$ and $f, g: \mathbb{C} \longrightarrow \mathbb{C}$ are two continuous functions satisfying *Unit 7*, Section 7.3

$$|f(P)| < |g(P)|$$

for all points P on the loop l. Then the winding numbers about 0 satisfy

$$\omega(g(l)) = \omega(h(l)), \qquad \text{where } h \text{ is the sum function } f + g.$$

Unit 8 Poincaré Index Theorem

Flow field = vector field A function V which assigns to each point (x, y) of $\mathbb{R}^2$ a vector $V(x,y)$ based at (x, y). In coordinates, we write *Unit 8*, Section 8.1

$$V(x, y) = (f(x, y), g(x, y)).$$

We assume that the functions f, g have continuous partial derivatives throughout this unit, without stating it every time.

Linear flow field The functions f, g are linear functions *Unit 8*, Section 8.1

$$f(x, y) = ax + by, \quad g(x, y) = cx + dy$$

(a, b, c, d real numbers).

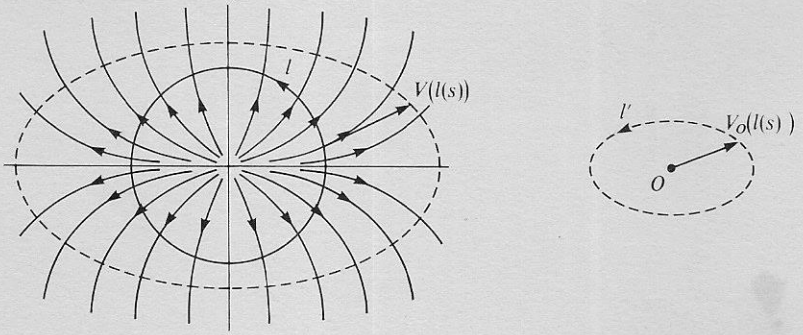
Rest point = singular point = singularity A point where $V(x, y) = (0, 0)$.

Unstable rest point A rest point such that any point close to the rest point "flows away" to infinity as t increases.

Stable rest point A rest point such that any point close to the rest point remains close as t increases.

Isolated rest point = isolated singular point A rest point so that no other nearby point is a rest point.

Index of an isolated singular point Draw a small simple loop l (say the boundary of a small disc), centre P, which winds once positively about P. At each point $l(s)$ of the loop, there is therefore a non-zero flow vector $V(l(s))$. Let $V_O(l(s))$ be a vector equal to $V(l(s))$, but based at some fixed point O. The "sharp end" of $V_O(l(s))$ will trace out a loop l' as $l(s)$ traces out l. The index of P is the winding number of l' about O. *Unit 8*, Section 8.1



This definition allows P to be a non-singular point where $V(P) \neq 0$—the index is then zero—but it does *not* allow P to be a point on a line of singular points. *Unit 8*, Section 8.1

Theorem (Poincaré Index Theorem) Let S be any closed orientable surface, and V a flow on S with only isolated singularities. Then the sum of the indices of all singular points of V is equal to $\chi(S)$.

Corollaries

1. The torus is the only closed orientable surface which has a smooth flow without singular points.
2. (Hairy Ball Theorem)
   There is no non-singular flow on S^2.

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